Efficient Delta-Parametrization of 2D Surface-Impedance Solutions

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Impedance boundary condition methods (IBCs) are among the most efficient methods for solving time-harmonic eddy-current problems with a small skin depth (delta). However for a wide range of frequencies (or material conductivities) the standard approach is no more efficient, since it requires for each frequency (or conductivity) the computation of a finite element (FE) complex-valued problem. Moreover, the accuracy of IBC decreases dramatically for large delta. As an extension of our previous work, we propose here a more detailed method of parametrization in delta of the 2D small-delta eddy-currents problem. This numerically efficient method gives a very good precision for all the frequencies difficult to address, i.e. from the frequency corresponding to the last good solution obtainable by meshing the conductor up to infinity (perfect conductor solution).

Index Terms—**Surface impedance, parametric solutions, small skin depth.**

I. Introduction

The classical surface impedance method makes it possible to solve approximately and quite accurately a time-harmonic eddy-current problem in a conductor (with a linear magnetic behavior), when the skin depth δ is small compared to the characteristic size D of the conducting parts Ω_C of the device [1]. If the boundary Σ of the conductor is regular enough, one can compute the electromagnetic field in the outer domain Ω_O by imposing a surface impedance condition on Σ, i.e.:

$$\mathbf{curl\ H} = \mathbf{J}_s \text{ in } \Omega_O, \quad (1)$$

$$\mathbf{n} \times \mathbf{E} = \underline{Z}_S\, \mathbf{n} \times (\mathbf{n} \times \mathbf{H}) \text{ on } \Sigma, \quad (2)$$

$$\underline{Z}_S = (1+j)/(\sigma\delta), \quad (3)$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{J}_s$ the source current density, $\mathbf{E}$ the electric field, $\mathbf{n}$ the outward normal, j the imaginary unit and $\underline{Z}_S$ the so-called surface impedance that depends on the electric conductivity σ and skin depth δ. The finite element solution is straightforward, e.g., in a 2D plane case, the vector potential ($\boldsymbol{A}$) formulation gives ($\boldsymbol{A}$ and $\mathbf{J}$ with only one component):

$$-\Delta\underline{A} = \mu_0 J_s \text{ in } \Omega_O;\ \underline{A} = \underline{\alpha}\delta\, \partial_n \underline{A} \text{ on } \Sigma;\ \underline{\alpha} = (j-1)/2 \quad (4)$$

If the frequency (or conductivity) is modified, the solution has to be performed again.

Note that this “classical” IBC belongs to a hierarchy of more and more precise approximations of the physical eddy-currents problem. Order 0 is the “perfect conductor” solution (error in δ); order 1 is the classical surface impedance (1)–(3), also called “Leontovich condition”, with error in δ^2 for curved surfaces; order 2 takes the scalar curvature of the boundary into account (error in δ^3) and coincides with Leontovich for flat areas; for higher orders, differential operators are involved on the boundary [1].

II. Mathematical Asymptotic Expansion

Theoretically, the 2D solution of Problem (4) can be formally expanded in a series in power of ($\underline{\alpha}\delta$) as in [2]:

$$\underline{\tilde{A}}_n(\delta) = (\underline{\alpha}\delta)^0 A_0 + (\underline{\alpha}\delta)^1 A_1 + (\underline{\alpha}\delta)^2 A_2 + \cdots + (\underline{\alpha}\delta)^n A_n \quad (5)$$

where the coefficients A_i are real-valued solutions to the recurrent elementary problems (6)–(7) independent of δ:

$$-\Delta A_0 = \mu_0 J_s \text{ in } \Omega_O;\ A_0 = 0 \quad \text{on } \Sigma, \quad (6)$$

$$\forall i \geq 1,\ -\Delta A_i = 0 \quad \text{in } \Omega_O;\ A_i = \partial_n A_{i-1} \text{ on } \Sigma. \quad (7)$$

The “perfect conductor” solution corresponds to the first term A_0. The Leontovich condition corresponds exactly to the infinite development ((5) with $n\to\infty$); however, the limited development ($\underline{\tilde{A}}_1$) is as good as the Leontovich solution, because their errors with the solution of the physical problem (1)–(3) have the same order of magnitude in δ^2.

In other words, ($\underline{\tilde{A}}_1$) is an order-1 delta-parametrized solution, whose error magnitude is of the same order as the classical surface impedance solution. The two basis solutions A_0 and A_1 are easily obtained by solving two (real-valued, one domain, coarse mesh) problems {(6)–(7), i=1} in the outer domain Ω_O.

One may wonder if it is possible to improve the accuracy. We first tried to include more terms in (5)–(7), but on one hand the numerical stability of (7) is bad for i=2 (and catastrophic for i>2); while on the other hand, the theoretical error compared with the physical solution is not improved. Then another approach must be pursued/adopted/tested.

III. A Pragmatic Method

A. Purposes

Consider the finite element solution ($\underline{A}_{FE}$) of the complete magnetodynamic problem in $\Omega = \Omega_O \cup \Omega_C$, obtained with the finest “acceptable” mesh (in term of numerical cost, so called “coarse mesh”) of the conducting region Ω_C; f_{FE} and δ_{FE} are the corresponding frequency and skin depth (typically, δ_{FE} is 15-20% of the characteristic size D of Ω_C and we have 4 or 5 first order elements in δ_{FE} all along Σ, Fig. 1). We aim at proposing a parametrized solution $\underline{\hat{A}}(\delta)$ such that, in Ω_O:

$$\lim_{\delta\to 0}\{[\underline{\hat{A}}(\delta) - \underline{\tilde{A}}_1(\delta)]/\delta\} = 0 \ and\ \lim_{\delta\to\delta_{FE}} \underline{\hat{A}}(\delta) = \underline{A}_{FE}. \quad (8)$$

This way, a good precision is expected for any frequencies greater that f_{FE}; "good" means equivalent to the precision of the surface impedance method for very high frequencies, and better for lower frequencies, up to f_{FE} for which the precision will be that of the complete coarse mesh solution ($\underline{A}_{FE}$).

B. Proposed empirical method.

Based on an asymptotic expansion as (5) and on the expected low frequency limit ((8), $\delta \rightarrow \delta_{FE}$), we propose to consider the following 3rd order expansion in Ω_O:

$$\underline{\hat{A}}(\delta) = (\underline{\alpha}\delta)^0\hat{A}_0 + (\underline{\alpha}\delta)^1\hat{A}_1 + (\underline{\alpha}\delta)^2\hat{A}_2 + (\underline{\alpha}\delta)^3\hat{A}_3, \quad (9)$$

where ($\hat{A}_0$, $\hat{A}_1$)=(A_0, A_1) are given by the method in section II., using 2 finite element solutions in the outer domain Ω_O; and ($\hat{A}_2$, $\hat{A}_3$) are such that the expected limit is exactly reached in δ_{FE}:

$$\underline{\hat{A}}(\delta_{FE}) = \underline{A}_{FE} \text{ in } \Omega_O. \quad (10)$$

The computational cost to get the 2 real-valued potentials ($\hat{A}_2$, $\hat{A}_3$) is that of one FE (coarse meshed) complex-valued solution in the complete domain Ω.

IV. Results for a Test Problem

The test problem (Fig. 1) in this digest is the one described in [3]: we enforce a flux on part of the boundary Σ of a conducting rounded angle. This is the simplest possible problem with flat and curved parts for Σ (note that the question of parametrized solution for eddy currents near corners was previously tackled by the same group of authors [4], a further work will combine the proposed techniques). Fig. 2 presents the profiles of the 4 basis potentials ($\hat{A}_0 \ldots \hat{A}_3$).

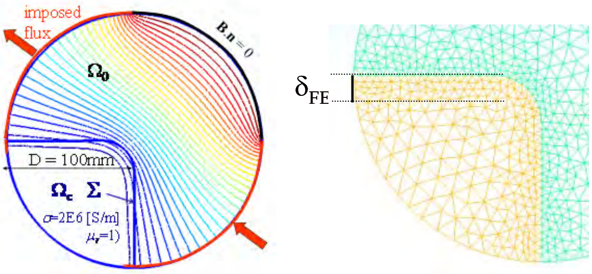

Fig. 1. The test problem (left) and a typical coarse mesh (δ_{FE}=15%*D*).

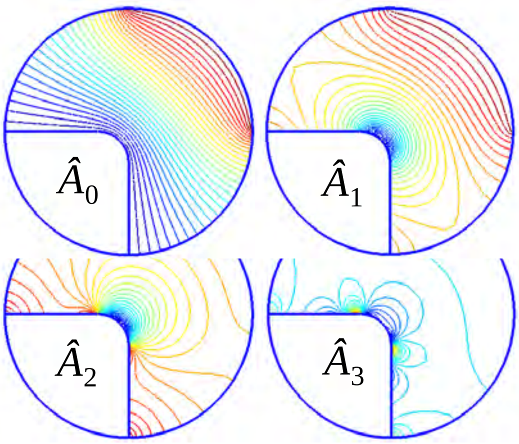

Fig. 2. Behaviors of the 4 basis solutions (9)–(10) obtained for δ_{FE}=15%*D*.

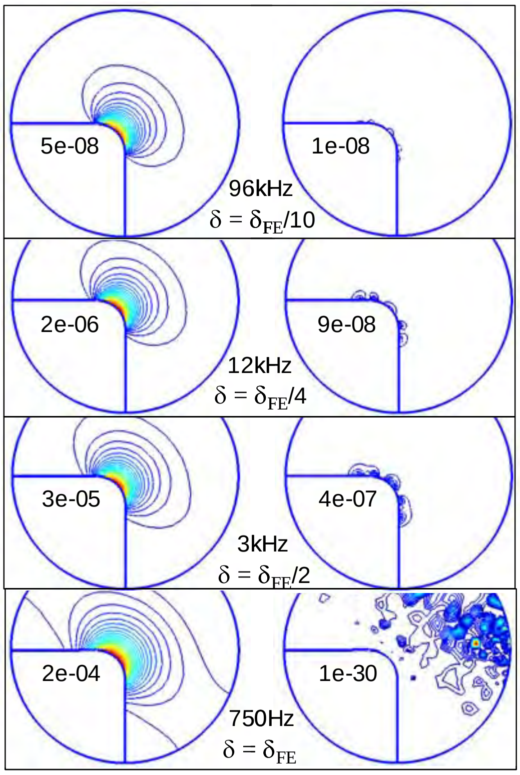

Fig. 3. Quadratic solution errors : IBC (left); proposed method (right).

The quadratic error of the proposed method is compared (Fig. 3) to the error of the classical IBC condition, for frequencies covering the range [f_{FE},∞[.

Conclusion and Perspectives

The computation cost of the proposed method is less than two IBCs solutions, regardless of the number of frequencies to consider. This clearly shows that we achieved our goal, in terms of precision and computation costs.

The extended paper will present more complex examples and preliminary tests for 3D structures. Future work will investigate the coupling of this parametrization technique with ideas previously proposed [4] for conductors with corners as well as rigorous justification of the accuracy.

References

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JE SUIS CHARLIE

In memory and sympathy of Jan. 7, 2014 victims in Paris.
Murdered for fighting for the freedom of expression and against fundamentalism/obscurantism/extremism with their brains and pens.